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Dominant Eigenvalue of a Sudoku Submatrix

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Abstract

Sudoku is a puzzle game that can be viewed in terms of square matrices. In particular, we will find that largest possible eigenvalue for a 3×3 Sudoku submatrix is $\sqrt{285}$. This will be found using the Gershgorin Circle Theorem as well as matrix norms.

1 Introduction

The first Sudoku puzzle game was created by a retired architect named Howard Garns and appeared in the May 1979 edition of *Dell Pencil Puzzles and Word Games*. The puzzle got its name when it appeared in a Japanese magazine in 1984. The Japanese name of "Sudoku" is loosely translated as "single numbers." Delahaye (2006)

The standard Sudoku puzzle consists of 9 rows, 9 columns, and 9 Sudoku submatrices, or 3×3 blocks. To start, there will be numbers that are pre-filled into the puzzle's cells. To solve the puzzle, one must fill in every cell of the 9×9 Sudoku only using numbers 1 to 9. This must be done so that no number is repeated in the respective row, column, and Sudoku submatrix.

In this paper, we will discuss the structure of Sudoku puzzles to prove that the dominant eigenvalue, or maximum possible eigenvalue, of any 3×3 Sudoku submatrix is at most $\sqrt{285}$. We will use the Gershgorin Theorem to find the upper bound for the eigenvalues. Then, matrix norms will be used to determine the highest possible eigenvalue of a Sudoku submatrix.

Examples of a given Sudoku and its solution are given below.

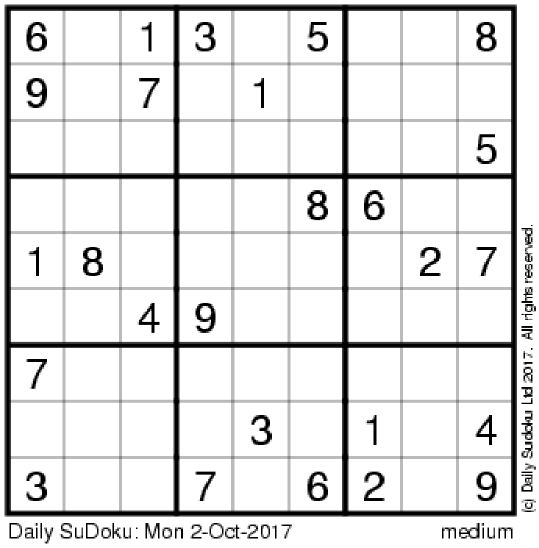

Figure 1: Example Sudoku given without a solution Daily Sudoku (2017a)

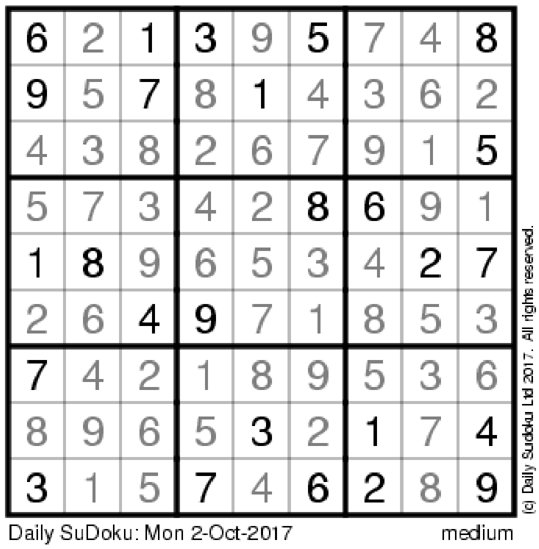

Figure 2: Example Sudoku with solution Daily Sudoku (2017b)

As stated, the paper will focus on the 3×3 Sudoku submatrices and the formal definition of the submatrix given below.

Definition 1.1. *A Sudoku submatrix is an $n \times n$ array with positive real values ranging from 1 to n^2, with each number appearing exactly once.*

For the 3×3 Sudoku submatrix, each element, 1 to 9, appears exactly once within the submatrix.

2 Background

In 2010, Merciadri Luca conducted research Sudoku Matrices by comparing properties of Sudoku submatrices with Sudoku matrices. The paper offers a partial proof that the dominant eigenvalue of a Sudoku submatrix is less than or equal to 22. Luca came close to proving this finding using maximum column sums and maximum row sums. Luca (2010)

3 Results

3.1 Gershgorin Circle Theorem

In this section, we will discuss what the Gershgorin Circle Theorem is and how it relates to the eigenvalues of Sudoku submatrices. The Gershgorin Circle Theorem, otherwise known as the Gershgorin Disc Theorem, is used to find the bounds of a complex square matrix's eigenvalues. Since Sudoku submatrices are square and each element is a real number, which is a subset of the complex numbers, the theorem can be applied to Sudoku submatrices.

The Gershgorin Circle Theorem uses the diagonal entries of complex square matrix and the nondiagonal entries in each respective i^{th} row to create Gershgorin Discs. The eigenvalues will fall within these Gershgorin Discs.

Theorem 3.1. *Let A be an $n \times n$ complex square matrix and let D_i for $i=$ 1,2,3,...,n be the closed disc, also known as the Gershgorin Disc, centered around A_{ii} with radius given by the row sum $r_i = \sum_{j \neq i} \mid A_{ij} \mid$, or the sum of the nondiagonal entries in the respective i^{th} row;*

$$D_i = \{z \in \mathbb{C} : \mid z - A_{ii} \mid \leq r_i\} = B(a_{ii}, \sum_{j=1, j \neq i}^{n} \mid a_{ij} \mid)$$

$B = A - \lambda I$ *and* I *is the identity matrix.*

Every eigenvalue of A lies in some D_i.

Proof. Let λ be an eigenvalue of A. We know that $B = A - \lambda I$, where I is the identity. That is $B = (b_{ij})$, with $b_{ii} = a_{ii} - \lambda$ and $b_{ij} = a_{ij}$, for $i \neq j$. Since $det(B) = det(A - \lambda I) = 0$, then the rows of B are linearly dependent. We can assume that $B_{r+1} = \alpha_1 B_1 + \alpha_2 B_2 + ... + \alpha_r B_r$, where B_i is the i^{th} row of $B, r+1 \leq n$ and $\alpha_i \in \mathbb{C}$ for $i = 1, 2, ...r$. So

$$B_{r+1} = (\sum_{k=1}^{r} \alpha_k b_{k,1}, \sum_{k=1}^{r} \alpha_k b_{k,2}, ..., \sum_{k=1}^{r} \alpha_k b_{k,n}).$$

If for some k between 1 and r, $|b_{k,k}| \leq \sum_{i=1,i\neq k}^{r+1} |b_{k,i}|$ then λ lies in D_k.

Now let's consider the other case. Suppose that $|b_{k,k}| > \sum_{i=1,i\neq k}^{r+1} |b_{k,i}|$, for every $k \in [1, 2, ...r]$, then $|\alpha_k b_{k,k}| \geq \sum_{i=1,i\neq k}^{r+1} |\alpha_k b_{k,i}|$, thus $|\alpha_k b_{k,r+1}| \leq -\sum_{i=1,i\neq k}^{r} |\alpha_k b_{k,i}| + |\alpha_k b_{k,k}|$. This inequality is for $\alpha_k \neq 0$, for some $k \in [1, 2, ...r]$.

Therefore,

$$
\begin{aligned}
\mid b_{r+1,r+1} \mid &= \mid \sum_{k=1}^{r} \alpha_k b_{k,r+1} \mid \\
&\leq \sum_{k=1}^{r} \mid \alpha_k b_{k,r+1} \mid \\
&< \sum_{k=1}^{r} \mid \alpha_k b_{k,k} \mid - \sum_{i=1,i\neq k}^{r} \mid \alpha_k b_{k,i} \mid \\
&= \sum_{k=1}^{r} \mid \alpha_k b_{k,k} \mid - \sum_{i=1,i\neq k}^{r} (\sum_{i=1,i\neq k}^{r} \mid \alpha_k b_{k,i} \mid) \\
&= \sum_{i=1}^{r} \mid \alpha_i b_{i,i} \mid - \sum_{i=1}^{r} (\sum_{k=1,k\neq i}^{r} \mid \alpha_k b_{k,i} \mid) \\
&= \sum_{i=1}^{r} (\mid \alpha_i b_{i,i} \mid - \sum_{k=1,k\neq i}^{r} \mid \alpha_k b_{k,i} \mid) \\
&\leq \sum_{i=1}^{r} (\mid \alpha_i b_{i,i} \mid - \mid \sum_{k=1,k\neq i}^{r} \alpha_k b_{k,i} \mid) \\
&\leq \sum_{i=1}^{r} \mid \alpha_i b_{i,i} + \sum_{k=1,k\neq i}^{r} \alpha_k b_{k,i} \mid \\
&= \sum_{i=1}^{r} \mid \sum_{k=1}^{r} \alpha_k b_{k,i} \mid \\
&= \sum_{i=1}^{r} \mid b_{r+1,i} \mid \\
&\leq \sum_{i=1,i\neq r+1}^{r} \mid b_{r+1,i} \mid,
\end{aligned}
$$

since $|a+b| \geq |a| - |b|$. It also should be noted that in line 5 of the summations above, the "i" summation was factored out, which switched the summation variables.

The summations above show that even when$|b_{k,k}| > \sum_{i=1,i\neq k}^{r+1} |b_{k,i}|$, $|b_{r+1,r+1}| \leq \sum_{i=1,i\neq r+1}^{r} |b_{r+1,i}|$. This shows that the eigenvalues are then lie within the row sum of the matrix. Thus, every eigenvalue lies in some D_i^{r+1}. Because this is the $r+1$ case, we see that can generalize this for any row i, showing that the eigenvalues lie within D_i. Gomez (2006)

$\square$

Example 3.2. *We are given a completed Sudoku submatrix, A, below.*

$$\begin{bmatrix} 9 & 1 & 8 \\ 2 & 4 & 3 \\ 7 & 5 & 6 \end{bmatrix}$$

To find the eigenvalues, set $det(A - \lambda I) = 0$ *and solve for the three* λs. *This matrix has three corresponding eigenvalues,* λ_i, *given below.*

$$\lambda_1 \approx 16.27740$$

$$\lambda_2 \approx 3.63524$$

$$\lambda_3 \approx -0.19126$$

The diagonal entries will be plotted on the complex plane as the centers of the three Gershgorin Discs for each respective row. Because all entries for a Sudoku *submatrix are real numbers, the center for each disk will be plotted on the horizontal axis, otherwise known as the real axis on the complex plane. In this case, the centers are (9,0), (4,0), and (6,0), as seen on the matrix's main diagonal.*

In each row, the summations of the absolute values of the non-diagonal entries in each respective row are used to find the radius of the Gershgorin Discs.

$$r_1 = |1| + |8| = 9$$

$$r_2 = |2| + |3| = 5$$

$$r_3 = |7| + |5| = 12$$

We have the three Gershgorin Discs, D_i with the properties below.

$$D_1 : center = (9,0); r_1 = 9$$

$$D_2 : center = (4,0); r_2 = 5$$

$$D_3 : center = (6,0); r_3 = 12$$

Plotting both the Gershgorin Discs and the eigenvalues on the complex plane, we obtain the graph below.

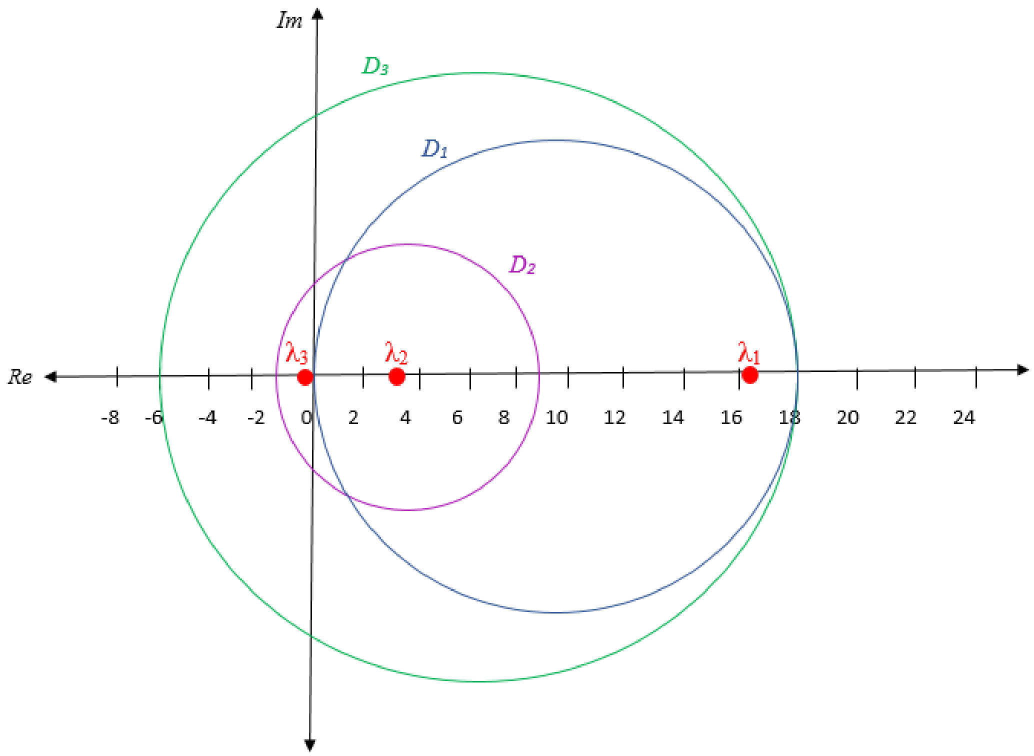

Figure 3: D_1=Blue; D_2=Purple; D_3=Green; Eigenvalues=Red

As shown in the figure above, all the eigenvalues are bounded by the Gershgorin Discs.

Using the Gershgorin Circle Theorem, we realize that the maximum eigenvalue of a 3×3 Sudoku submatrix is less than or equal to 24. Any row containing any order of [7,8,9], which are the highest possible elements, will create a respective Gershgorin Disc that will reach an upper bound of 24. This is the largest upper bound that can be reached because the row contains the largest three values.

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For example, if a Sudoku submatrix contains [7,8,9] in the same i^{th} row with 7 as the diagonal entry, its associated D_i would have a radius of $|8| + |9| = 17$ and a center of (7,0), which would create a disc with an upper bound of 24 as shown below.

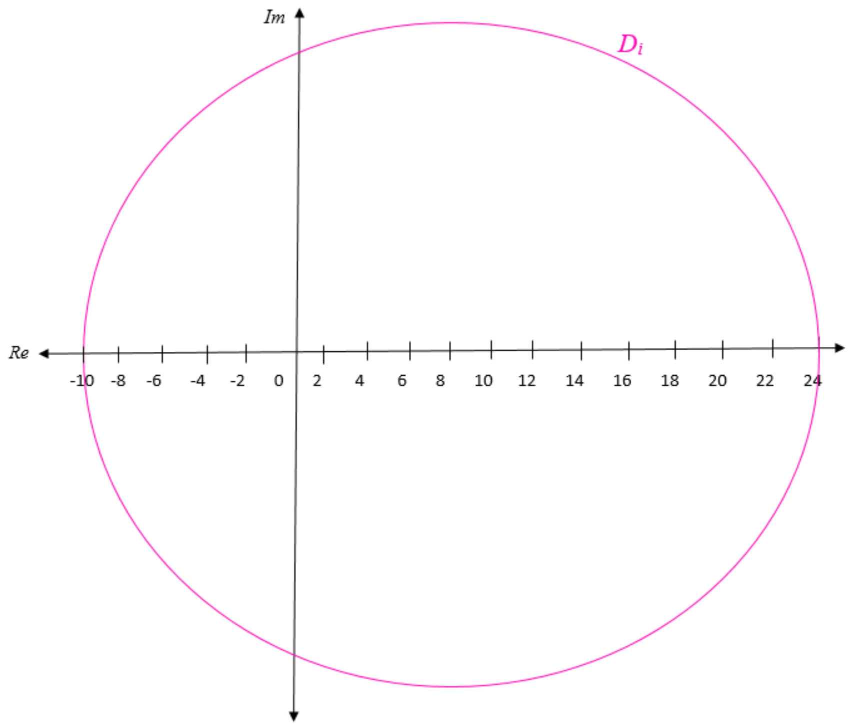

Figure 4: D_i = Pink

If a different Sudoku submatrix contained [8,7,9] in the same i^{th} row with 8 as a diagonal entry, its associated D_i would have a radius of $|7| + |9| = 16$ and a center of (8,0), which would create a disc with an upper bound of 24 as shown below.

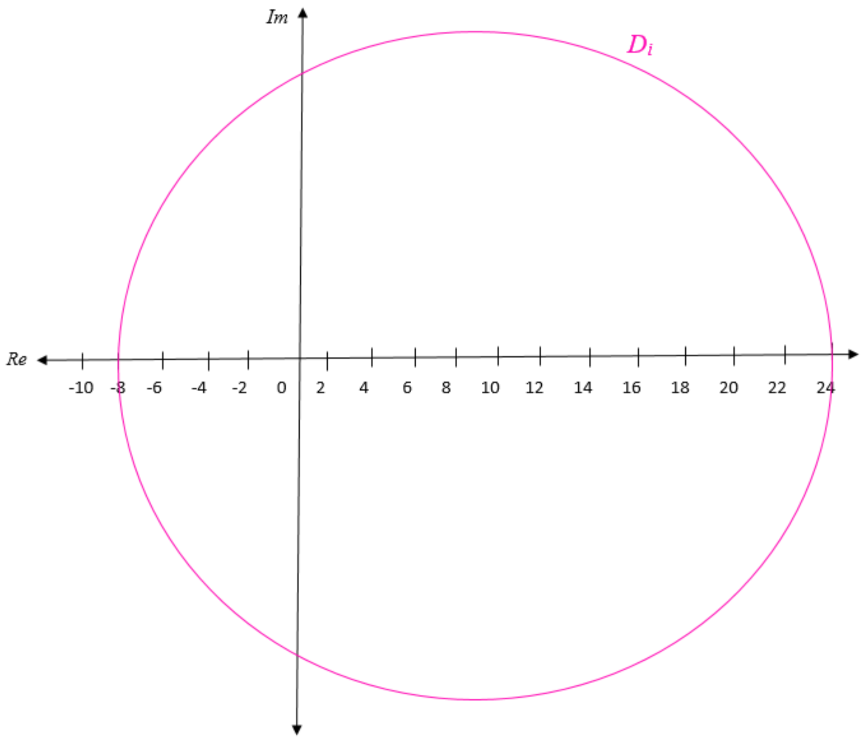

Figure 5: D_i = Pink

Similarly, if a different Sudoku submatrix contained [9,8,7] in the same i^{th} row with 9 as a diagonal entry, its associated D_i would have a radius of $|7| + |8| = 15$ and a center of (9,0), which would create a disc with an upper bound of 24 as shown below.

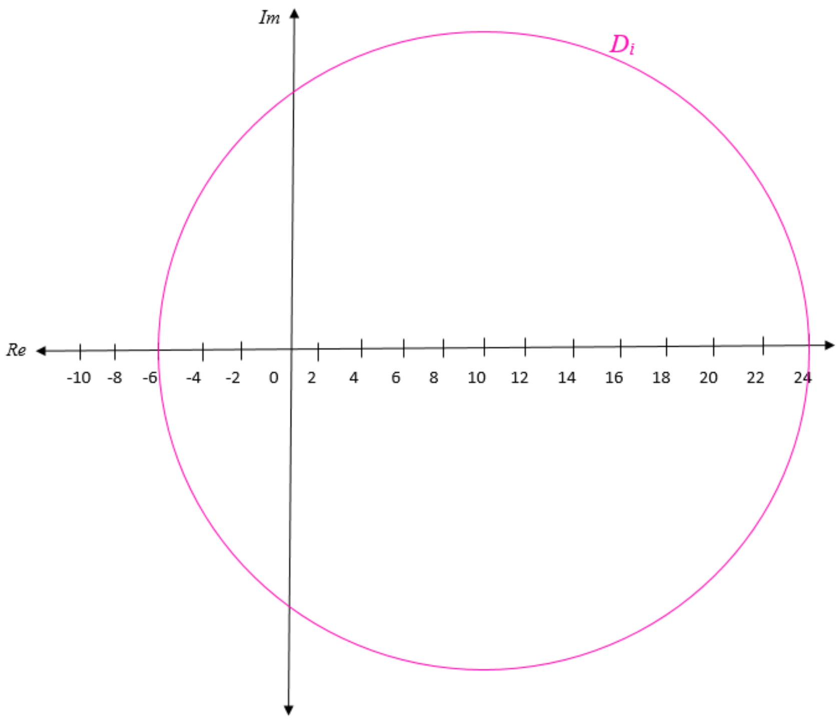

Figure 6: D_i = Pink

However, the maximum eigenvalue may never hit 24 for a 3×3 Sudoku submatrix. The Gershgorin Theorem does not guarantee that the maximum eigenvalue can reach 24. The Gershgorin Discs only provide bounds for the eigenvalues. More information needed to find the maximum eigenvalue that is attainable.

3.2 Matrix Norms

In this section, we will find that matrix norms of a 3×3 Sudoku submatrix tell us more about the maximum eigenvalue of a Sudoku submatrix. These will provide us with an attainable maximum possible eigenvalue.

Definition 3.3. *Given a square complex or real matrix, A, a matrix norm , $||A||$, is a nonnegative number associated with A having the properties:*

1. $||A|| > 0$ *when* $A \neq 0$ *and* $||A|| = 0$ *if and only if* $A = 0$.
2. $||kA|| \ = \ |k| \ \ ||A||$ *for any scalar* k.
3. $||A + B|| \ \leq \ ||A|| + ||B||$
4. $||AB|| \leq ||A|| \ \ ||B||$

Weisstein (2017a)

Definition 3.4. *Given a vector, x, a vector norm,$||x||$, is a nonnegative number defined such that:*

1. $||x|| > 0$ *when* $x \neq 0$ *and* $||x|| = 0$ *iff* $x = 0$.
2. $||kx|| = |k|||x||$ *for any scalar k.*
3. $||x + y|| \leq ||x|| + ||y||$

Weisstein (2017b)

We will be focusing on three different types of matrix norms, the Maximum Absolute Column Sum Norm, the Spectral Norm, and the Maximum Absolute

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Row Sum Norm. Each of the following matrix norms are induced norms, meaning that they can be written in terms of their respective vector norms.

Definition 3.5. *The Maximum Absolute Column Sum Norm, $||A||_1$, is defined as $||A||_1 = max_j(\sum_{i=1}^{n} |a_{ij}|) = max\frac{||Ax||_1}{||x||_1}$ where $||x||_1 \neq 0$.*

Definition 3.6. *The Spectral Norm, $||A||_2$, is defined as $||A||_2 =$ (maximum eigenvalue of $A^T A)^{1/2} = max\frac{||Ax||_2}{||x||_2}$ where $||x||_2 \neq 0$.*

Definition 3.7. *The Maximum Absolute Row Sum Norm $||A||_\infty$ is defined as $||A||_\infty = max_i(\sum_{j=1}^{n} |a_{ij}|) = max\frac{||Ax||_\infty}{||x||_\infty}$ where $||x||_\infty \neq 0$.*

Weisstein (2017a)

To see how each induced matrix norm is calculated, let us consider an example.

Example 3.8. *We are given a complete 3×3 Sudoku submatrix A below.*

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

In this case,

$||A||_1 = 18$ because the largest absolute column sum is $|3| + |6| + |9| = 18$.

$||A||_\infty = 24$ because the largest absolute row sum is $|7| + |8| + |9| = 24$.

To find $||A||_2$ we need to find the transpose of the matrix above. A^T is given below.

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

After multiplying $A^T A$, we arrive at the matrix below.

$$\begin{bmatrix} 66 & 78 & 90 \\ 78 & 93 & 108 \\ 90 & 108 & 126 \end{bmatrix}$$

We can now find that the maximum eigenvalue of $A^T A$ is approximately 283.859. Given this, $||A||_2 = \sqrt{283.859} \approx 16.848$.

By definition, eigenvalues are related to a square matrix and a corresponding vector as shown below.

Definition 3.9. *Given a square complex matrix A, λ is an eigenvalue of A if there is a non-zero vector $u \in \mathbb{C}$ such that $Au = \lambda u$.*

We will use this definition to connect the maximum eigenvalue of a square matrix to each of the induced matrix norms.

Theorem 3.10. *The maximum eigenvalue of an $n \times n$ matrix is A is less than or equal to $||A||$, where $||A||$ is any of the induced norms.*

Proof. Let λ be any eigenvalue of A with an eigenvector of x.

$|\lambda| = |\lambda|||x|| = ||\lambda x|| = ||Ax|| \leq ||A||||x|| = ||A||$

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So, $|\lambda| \leq ||A||$. $\square$

Based on this theorem, the goal to find the maximum eigenvalue of a 3×3 Sudoku submatrix is to maximize each of the induced norms.

We can see that $||A||_\infty$ and $||A||_1$ are each maximized at 24, but never at the same time. This is because [7,8,9] are the only combination of elements whose sum will equal 24. Because no number from 1 to 9 can appear more than once, it is only possible for one row sum or one column sum to equal 24. We see that $||A||_\infty$ and $||A||_1$ are separately maximized at 24.

However, maximizing the Spectral Norm, $||A||_2$, is not as direct as adding up elements within the matrix. We can start by focusing on the relation between a square matrix and its transpose. By doing so, we will see how this maximizes $||A||_2$ and affects the maximum possible eigenvalue.

Theorem 3.11. *The eigenvalues of A^T are equal to the eigenvalues of A.*

Proof. The eigenvalues of an $n \times n$ matrix A, known as λ_A are the values when $\det(A - \lambda I) = 0$. Now let's consider matrix A^T.

$$\begin{aligned}(A^T - \lambda I) &= (A - \lambda I)^T \\ &= (A - \lambda I)\end{aligned}$$

This equality is possible because the determinant of a square matrix is the same as the determinant of its transpose. Therefore, the eigenvalues of A and A^T are equivalent. $\square$

Since we know the eigenvalues of a matrix A are equal to the eigenvalues of A^T, we can now connect this to Sudoku submatrices. The transpose of a 3×3 Sudoku submatrix creates another Sudoku submatrix with the same eigenvalues. So, with the same eigenvalues, the matrix and its transpose have the same maximum eigenvalue. It follows that a Sudoku submatrix will have the same eigenvalue bounds as its transpose.

Since we have proved that the maximum eigenvalue of a matrix is less than or equal to any of its induced norms, it must follow that these bounds hold for the transpose of the matrix. In other words, the induced norms of a matrix must remain the same for its transpose because it represents an upper bound for the maximum eigenvalue.

Let's see how matrix norms react to the transpose of a Sudoku submatrix. We are given a completed 3×3 Sudoku submatrix A. It is always true that $||A||_\infty = ||A^T||_1$ and $||A||_1 = ||A^T||_\infty$. However, $||A||_\infty$ does not always equal $||A^T||_\infty$ unless $||A||_\infty = ||A||_1$. Similarly, $||A||_1$ does not always equal $||A^T||_1$ unless $||A||_1 = ||A||_\infty$. This is because the highest row sum turns into the highest column sum when taking the transpose of a matrix. The reverse is also true when taking the transpose of a matrix.

On the other hand, $||A||_2 = ||A^T||_2$.

We will denote p as the maximum eigenvalue of a matrix. $||A||_2 = \sqrt{p(A^TA)} =$

$\sqrt{p(AA^T)} = ||A^T||_2$. So we see that the Spectral Norm still holds that a matrix and its transpose have the same bounds, however only the Spectral Norm will stay the same when taking the transpose.

We see this in the example below.

Example 3.12. *We are given a Sudoku submatrix, A, below.*

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$||A||_1 = 18, ||A||_\infty = 24, ||A||_2 \approx 16.848$

The transpose of A, A^T is shown below.

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$||A^T||_1 = 24, ||A^T||_\infty = 18, ||A^T||_2 \approx 16.848$

We see that $||A||_\infty = ||A^T||_1, ||A||_1 = ||A^T||_\infty$, and $||A||_2 = ||A^T||_2$.

Using the knowledge of how the transpose of a 3×3 Sudoku submatrix affects the induced matrix norms, we can prove that all of the induced norms are related to one another.

Theorem 3.13. *Given a square matrix A, $||A||_2^2 \leq ||A||_1||A||_\infty$. Weisstein (2017a)*

Proof. We start with defining $||A||_2^2$ as $p(A^TA)$. Where p represents the dominant eigenvalue of a matrix. We've proven that $p(A) \leq ||A||$.

The maximum eigenvalue of A^TA is less than or equal to $||A^TA||_1$ and $||A^TA||_\infty$.

$||A^TA||_1 \leq ||A^T||_1||A||_1$ and $||A^TA||_\infty \leq ||A^T||_\infty||A||_\infty$ using a property of matrix norms.

$||A^T||_1 = ||A||_\infty$ and $||A^T||_\infty = ||A||_1$.

Therefore, $p(A^TA) = ||A||_2^2 \leq ||A^TA||_1 \leq ||A||_1||A||_\infty$.

Similarly, $p(A^TA) = ||A||_2^2 \leq ||A^TA||_\infty \leq ||A||_1||A||_\infty$.

It follows that $||A||_2^2 \leq ||A||_1||A||_\infty$. $\square$

Utilizing the proven equation, $||A||_2^2 \leq ||A||_1||A||_\infty$, we can now maximize the Spectral Norm for some 3×3 Sudoku submatrix A using $||A||_1$ and $||A||_\infty$.

Let us consider maximizing $||A||_1$ or $||A||_\infty$ at 24.

For a 3×3 Sudoku submatrix, a row or column containing [7,8,9] is the only way to reach 24 as a row or column sum. So, exactly one row or column can add up to 24.

The next possible highest row or column sum within the same Sudoku submatrix is 20 using the next highest elements in a row or column of [5,6,9], as shown in the example below.

$$\begin{bmatrix} 9 & 8 & 7 \\ 5 & 1 & 2 \\ 6 & 3 & 4 \end{bmatrix}$$

Plugging this into the Spectral Norm inequality, $||A||_2 \leq \sqrt{||A||_1||A||_\infty} = \sqrt{(24 \times 20)} \approx 21.919$.

The next highest possible row or column sum is 23, which can only appear with the elements [6,8,9] present in the same row or column.

The next highest possible row or column sum within the same 3×3 Sudoku submatrix would be 21, with elements [5,7,9], as shown in the example below.

$$\begin{bmatrix} 9 & 8 & 6 \\ 7 & 1 & 2 \\ 5 & 3 & 4 \end{bmatrix}$$

Plugging this into the Spectral Norm inequality, $||A||_2 \leq \sqrt{||A||_1||A||_\infty} = \sqrt{(23 \times 21)} \approx 21.977$.

The next highest possible row and column sum is 22. However, this can be expressed in 2 ways, with elements [6,7,9] and [5,8,9] in rows or columns of the same Sudoku submatrix. A possible arrangement is shown below.

$$\begin{bmatrix} 9 & 8 & 5 \\ 7 & 1 & 2 \\ 6 & 3 & 4 \end{bmatrix}$$

Plugging this into the Spectral Norm inequality, $||A||_2 \leq \sqrt{||A||_1||A||_\infty} = \sqrt{(22 \times 22)} = 22$.

If we were to continue this trend for lower possible row and column sums, $||A||_2$ would always fall less than 22.

We then see that he Spectral Norm is maximized at 22 for an individual 3×3 Sudoku submatrix. This occurs when both $||A||_1$ and $||A||_\infty$ equal 22 on the same matrix.

We will now generalize this finding to all Sudoku submatrices. In other words, each of the maximized norms do not occur on the same matrix, but are generalized to *all* 3×3 Sudoku submatrices, represented by S. We have found the maximum possible values for each of the induced matrix norms that do not necessarily appear on the same Sudoku matrix as listed below.

The highest possible value for $||S||_1$ is 24. The highest possible value for $||S||_\infty$ is also 24. The highest possible value for $||S||_2$ is 22.

With a generalized S representing all 3×3 Sudoku submatrices, $p(S)$ must be less than or equal to its induced norms, so we conclude the following.

$$p(S) \leq max(||S||_1) = 24$$

and

$$p(S) \leq max(||S||_\infty) = 24$$

and

$$p(S) \leq max(||S||_2) = 22$$

Using Theorem 3.10, we see that $p(A) \leq ||A|| \leq 22$.

To see if the maximum eigenvalue of a 3×3 Sudoku submatrix can actually reach 22, we will consider a non-induced matrix norm.

Definition 3.14. *For a matrix A, the Frobenius Norm $||A||_F$ is defined as $||A||_F = \sqrt{trace(A^TA)}$ where the trace of an $n \times n$ matrix is the sum of the main diagonal values.*

This type of norm is not an induced matrix norm.

Because the Frobenius Norm is not an induced norm, it cannot directly relate to the maximum eigenvalue in the same way that an induced norm can (i.e. $p(A) \leq ||A||$). Meaning, the maximum eigenvalue of a square matrix is not necessarily less than or equal to its Frobenius Norm.

The Frobenius Norm can be calculated from an $n \times n$ matrix, so it can be applied to a Sudoku submatrix.

Theorem 3.15. *For any* 3×3 *Sudoku submatrix,* A*,* $||A||_F = \sqrt{285}$*.*

Proof. First, let's consider a 3×3 Sudoku submatrix A with elements $a, b, c, d, e, f, g, h, i$ representing distinct integers from 1 to 9.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The transpose of this matrix, A^T is

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

When multiplying A^TA, we get

$$\begin{bmatrix} aa+bb+cc & ad+be+cf & ag+bh+ci \\ ad+be+cf & dd+ee+ff & dg+eh+fi \\ ag+bh+ci & dg+eh+fi & gg+hh+ii \end{bmatrix}$$

The trace of A^TA is then equal to $aa+bb+cc+dd+ee+ff+gg+hh+ii$ because it is the sum of the main diagonal entries. This also shows a more generalized statement that the trace of a 3×3 matrix multiplied by its transpose is equal to the sum of every squared element.

So, for any 3×3 Sudoku submatrix the $trace(A^TA) = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 = 285$. That is because the elements must always range from 1 to 9 appearing exactly once. It then follows that $||A||_F = \sqrt{trace(A^TA)} = \sqrt{285}$. $\square$

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Although the Frobenius Norm does not directly relate to $p(A)$, it does relate to the Spectral Norm.

Theorem 3.16. *The spectral norm is less than or equal to the Frobenius Norm.*

Proof. Let p represent the maximum eigenvalue for a square matrix.

We know that $||A||_2^2 = p(A^T A)$ and trace$(A^T A)$ is equal to the sum of all the squared elements of a square matrix (shown in previous proof).

$||A||_2^2 = p(A^T A) \leq ||A^T A||_\infty \leq trace(A^T A) = ||A||_F^2$

Similarly, $||A||_2^2 = p(A^T A) \leq ||A^T A||_1 \leq trace(A^T A) = ||A||_F^2$

It then follows that $||A||_2 \leq ||A||_F$. $\square$

Now we must maximize the Spectral Norm to bound the maximum eigenvalue of a 3×3 Sudoku submatrix.If the Spectral Norm must be at most the Frobenius Norm, then the Spectral Norm cannot be maximized at 22. It would be maximized at $\sqrt{285}$.

Plugging this into the Spectral Norm inequality, $||A||_2 \leq \sqrt{||A||_1 ||A||_\infty} = \sqrt{(15 \times 19)} = \sqrt{285} \approx 16.882$.

This occurs when either $||A||_1$ or $||A||_\infty$ must equal 19 and the other 15 on the same 3×3 Sudoku submatrix. The values of 15 and 19 does not constitute the generalized maximum possible value of $||A||_1$ and $||A||_\infty$ as either norm is maximized at 24 on separate matrices.

Consequently, we see that the Spectral Norm is maximized at $\sqrt{285}$ because the Frobenius Norm is the Spectral Norm's upper bound. This is true for every Sudoku submatrix. Generalizing to all 3×3 Sudoku submatrices, S, we maximize the induced norms using the Frobenius Norm as the Spectral Norm's upper bound. This will maximize the dominant eigenvalue of S.

So,

$$p(S) \leq max(||S||_1) = 24$$

and

$$p(S) \leq max(||S||_\infty) = 24$$

and

$$p(S) \leq max(||S||_2) = 16.882$$

It follows that $p(S) \leq 16.882$.

Thus, using the Gershgorin Circle Theorem and matrix norms, we have found that the maximum possible eigenvalue of a 3×3 Sudoku submatrix is $\sqrt{285} \approx 16.882$.

4 Further Research

There are two other topics for future research that would help further the results.

The first topic would study other non-induced norms that may relate to any of the induced norms, further representing eigenvalue bounds.

The second topic would be to study the relationship between vector norms and eigenvalues of a matrix. For example, one may find how the norms of the eigenvectors relate to the corresponding eigenvalues.

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